Topic — Linear Differential Equations with Constant Co-efficients

B.Sc( ) (Part II) Date —

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Study Material —

The General Differential Equation (Linear) of nth order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X \quad (1)$$

Here X and the Co-efficients $P_1, P_2, P_3 \cdots P_n$ are functions of x only.

If $P_1, P_2, \cdots P_n$ are Constants and X be the function of x only, then the above Equation is known as the linear differential Equation with Constant Co-efficients Equation.

The operator D :→ D for $\frac{d}{dx}$

D^2 for $\frac{d^2}{dx^2}$

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D^n for $\frac{d^n}{dx^n}$

Thus in term of operator D the Differential equation (1) can be written as

$$[D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n]y = X$$

Auxiliary Equation :→ $f(D) = 0$

Can in General be regarded as the Auxiliary Equation.

Solution of Equations:

When all the roots of Auxiliary Equation $f(D)=0$ are real and different—

If $m_1, m_2, m_3 \ldots m_n$ are n different real roots of $f(D)=0$

Then general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

Ex1: Solve $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0$

Solution — ~~Auxiliary Equation is~~ Equation is

$$(D^3 - 13D - 12)y = 0$$

Auxiliary Equation

$$D^3 - 13D - 12 = 0$$

i.e. $(D+1)(D+3)(D-4) = 0$

$\therefore D = -1, -3, 4$

Hence complete solution is

$$y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{4x}$$

Ex(2): Solve

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

Solution Given Equation is

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

Auxiliary Equation

$$D^3 + 6D^2 + 11D + 6 = 0$$

$\Rightarrow D^3 + D^2 + 5D^2 + 5D + 6D + 6 = 0$

$\Rightarrow D^2(D+1) + 5D(D+1) + 6(D+1) = 0$

$\Rightarrow (D+1)(D^2 + 5D + 6) = 0$

$\Rightarrow (D+1)[D^2 + 3D + 2D + 6] = 0$

$\Rightarrow (D+1)[D+3](D+2) = 0$

$\Rightarrow D = -1, -3, -2$

$\therefore$ Complete Solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

Case II - Auxiliary Equation having Equal roots :→

If the Auxiliary Equation has the real roots $m_1 \cdots m_k$ occurring k times and if further the remaining roots of the auxiliary Equation are distinct real numbers say $m_{k+1}, m_{k+2} \cdots m_n$

then $y = [c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}] e^{m_k x} + e^{m_{k+1} x} + \cdots e^{m_n x}$

Ex :→ $(D^3 - 3D + 2)y = 0$

Given Equation is

$$(D^3 - 3D + 2)y = 0$$

Auxiliary Equation

$$D^3 - 3D + 2 = 0$$

$\Rightarrow D^3 - D^2 + D^2 - D - 2D + 2 = 0$

$\Rightarrow D^2(D-1) + D(D-1) - 2(D-1) = 0$ $\qquad D^2 + D - 2 = 0$

$\Rightarrow (D-1)(D^2 + D - 2) = 0$ $\qquad \left[D = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \right.$

$\Rightarrow (D-1)(D-1)(D+2) = 0$ $\qquad = 1, -2$

$\Rightarrow (D-1)^2(D+2) = 0$ $\quad \therefore D = 1, 1, -2$ $\quad \therefore y = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$